Problem 1

Part a.

First off, Faraday's Law for electrostatics in differential point form is:

$$\nabla \times \mathbf{E} = 0 \tag{1}$$

We then take a surface integral over both side, the zero stays a zero, and apply Stoke's Theorem.

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{s} \tag{2}$$

We are now left with the integral form of Faraday's Law for electrostatics.

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{3}$$

Part b.

Now for a similar procedure for the Ampere's Law, starting in differential point form for magnetostatics.

$$(\nabla \times \mathbf{B}) = \mu \mathbf{J} \tag{4}$$

We now take a surface integral over both sides again, applying Stoke's Theorem on the left side.

$$\oint (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu \int \mathbf{J} \cdot d\mathbf{s} \tag{5}$$

We now use the following relation for the current and current density:

$$I = \int \mathbf{J} \cdot d\mathbf{s} \tag{6}$$

We are left with Ampere's Law for magnetostatics in integral form.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu I \tag{7}$$

Problem 2

The beginning point for this problem is the point form for Gauss' Laws.

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \tag{9}$$

We now take the volume integral of both sides of both equations

$$\iiint (\nabla \cdot \mathbf{B}) dV = 0 \tag{10}$$

$$\iiint (\nabla \cdot \mathbf{E}) dV = \iiint (\frac{\rho}{\epsilon}) dV \tag{11}$$

Now we use divergence theorem on the left hand sides and the fact the volume integral of a charge density is the total charge enclosed in the volume. Divergence Theorem in below.

$$\iiint (\nabla \cdot \mathbf{B}) dV = \oint \mathbf{B} \cdot d\mathbf{s} \tag{12}$$

We end up with the following equations below which are the integral form of Gauss's Laws

$$\oint \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon} \tag{13}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = 0 \tag{14}$$

Then, reverse these steps if you wish to recover the point forms.

Problem 3

The process for this problem is almost the same as last one. Take a volume integral of both sides of the equation. We pull the $\frac{\partial}{\partial t}$ out of the integral because we know what the volume integral of ρ is, and the integral is not with respect to time.

$$-\iiint (\nabla \cdot \mathbf{J}) dV = \frac{\partial}{\partial t} \iiint \rho dV \tag{15}$$

Now we apply divergence theorem and replace the integral on the right side with q.

$$-\oint \mathbf{J} \cdot d\mathbf{s} = \frac{dq}{dt} \tag{16}$$

Problem 4

In order to satisfy Gauss's Law, the divergence of the **B**-field will have to be 0. Note that this **B**-field is in spherical coordinates.

$$\nabla \cdot \mathbf{B} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 B_R) + \frac{1}{R \sin(\theta)} \frac{\partial}{\partial \theta} (B_\theta \sin(\theta)) + \frac{1}{R \sin(\theta)} \frac{\partial B_\phi}{\partial \phi} \tag{17}$$

Now, since there are no ϕ or θ components of **B**, those terms go to zero and then evaluating the derivative gives 0 with respect to R since the R^2 cancel out. So it does satisfy Gauss's Law. Now we for the next portion we need to find the curl of the **B**-field to determine the current density from Faraday's Equation. I am only going to include the terms involve B_R since we know the other terms are 0.

$$\nabla \times \mathbf{B} = a_\theta \frac{1}{R} \frac{1}{\sin(\theta)} \frac{\partial B_R}{\partial \phi} - a_\phi \frac{1}{R} \frac{\partial B_R}{\partial \theta} \tag{18}$$

Now, evaluating the math and remembering to divide by μ to get $\mathbf{J}$ we obtain the following.

$$\mathbf{J} = \frac{1}{\mu}\left[\frac{\cos(\phi)\cos(\theta)^2}{R^3\sin\theta}a_\theta + \frac{2\cos(\theta)\sin(\theta)\sin(\phi)}{R^3}a_\phi\right] \tag{19}$$

Problem 5

For the last problem, we have an **E**-field and we need to find the charge density so we enact Gauss's Law again and take the divergence of it. This field is in cylindrical coordinates. The cylindrical divergence is below.

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_\phi}{r\partial\phi} + \frac{\partial E_z}{\partial z} \tag{20}$$

Evaluating the divergence and arranging Gauss's Law for ρ we get the following

$$\rho = \epsilon(9r - \sin(\phi)) \tag{21}$$

Now we simply evaluate at the point [2,π,1] and the result is

$$\epsilon(18) \tag{22}$$