Problem 1

For this problem we start off by defining a few quantities:

$$\rho_l = \frac{Q}{L} \tag{1}$$

$$\therefore$$

$$dq = \rho_l dx' \tag{2}$$

We can now define the vector $\mathbf{R} - \mathbf{R}'$, which is the vector pointing to p from the differential charge, dq.

$$\mathbf{R} - \mathbf{R}' = -y\hat{a}_y - x'\hat{a}_x \tag{3}$$

Note that the primed vector $\mathbf{R}'$ is given here as positive, since our limits of integration will be taken as $-L$ to 0. Now, we determine the electric field through the summation of differential charge elements. The following equation is a general form in which we will substitute our vector into and differential charge element.

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_o} \frac{dq}{|\mathbf{R} - \mathbf{R}'|^3} (\mathbf{R} - \mathbf{R}') \tag{4}$$

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_o} \frac{\rho_l dx}{(x'^2 + y^2)^{3/2}} (-y\hat{a}_y - x'\hat{a}_x) \tag{5}$$

We can integrate the equation above from $-L$ to 0. The integral breaks up into 2 parts, one for each of the components of the electric field. Then the integrals can be looked up in a table and evaluated. The final result is then:

$$\mathbf{E} = \frac{\rho_l}{4\pi\epsilon_o} \left[\frac{-L\hat{a}_y}{y\sqrt{y^2 + L^2}} + \left(\frac{1}{|y|} - \frac{1}{\sqrt{y^2 + L^2}} \right) \hat{a}_x \right] \tag{6}$$

Problem 2

For this problem we will first find the relation between the volume charge density and the surface charge density based on the $\mathbf{E}$-field being 0 outside the sphere. We need to get each of the quantities in terms of q. The surface charge will be $+q$ and the volume charge will be $-q$, so then the enclosed charge is 0 when we are outside the sphere.

$$q = \rho_s 4\pi b^2 = \rho_s(\text{Area}) \tag{7}$$

$$-q = \rho_v(\text{Volume}) = \frac{4}{3}\pi(a^3)\rho_v \tag{8}$$

We now just have to equate these terms to get the relation between the 2 densities. The final relation is below.

$$\rho_v = \frac{-3\rho_s b^2}{a^3} \tag{9}$$

For the first region of $R < a$ we can form a spherical Gaussian surface inside the sphere and determine the field that way. This Gaussian surface gives the following equation taking into account the charge is now dependent on r:

$$\mathbf{E} = \frac{\rho_v(\frac{4}{3}\pi r^3)}{4\pi\epsilon_o r^2}\hat{a}_r = \frac{\rho_v r}{3\epsilon_o}\hat{a}_r \tag{10}$$

Now the next field calculation is much easier since the only charge enclosed in the spherical Gaussian surface is the total sphere, so it is the same as the middle term of equation (9) but with $r^3 \to a^3$.

$$\mathbf{E} = \frac{\rho_v a^3}{3\epsilon_o r^2}\hat{a}_r \tag{11}$$

Problem 3

Now for the third problem we can find the relation between the linear charge density and the volume charge density in a similar manner as problem 2, as in by setting the total charge to zero outside the object. (I've already cancelled the l's that would of been on each side)

$$\rho_l = -\rho_v \pi(c^2 - b^2) \tag{12}$$

We can assume that the **E**-field goes to 0 when inside a conductor which is from $a < r < b$. This is because the charges separate and there is an accumulation of charges on each surface. This gives insight into how to treat the conductor for later on. Now for the electric field inside the conductor ($r < a$) we can apply Gauss's Law and we get the following.

$$\mathbf{E}(2\pi r l) = \frac{(\rho_l)l}{\epsilon_o}\hat{a}_r \tag{13}$$

Now we arrange and simplify

$$\mathbf{E} = \frac{\rho_l}{2\pi r\epsilon_o}\hat{a}_r \tag{14}$$

Now for the electric field in for $b < r < c$ we now take into account the fact that the volume charge is still dependent on the radius, but we have the total line charge. So for our cylindrical Gaussian surface we get the following:

$$\mathbf{E} = \frac{\rho_l + \rho_v\pi(r^2 - b^2)}{2\pi r\epsilon_o}\hat{a}_r = \frac{\rho_v\pi(r^2 - c^2)}{2\pi r\epsilon_o}\hat{a}_r \tag{15}$$

The charge relation (11) was used for the last step to simplify further.

Problem 4

Now for this next problem we will do in Cartesian coordinates. We need to integrate over an area since we have a surface charge density and we know that the differential area element for area in polar cylindrical coordinates is the following, we define our charge in cylindrical but our vectors in Cartesian.

$$dA = r'dr'd\phi' \tag{16}$$

And now for the differential charge approach to the electric field which is the following equation where $dq = dA\rho_s$ for us.

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_o}\frac{dq}{|\mathbf{R}-\mathbf{R}'|^{\frac{3}{2}}}(\mathbf{R}-\mathbf{R}') \tag{17}$$

For this setup only the z component of the vector will be left and the radial one will cancel out. This reduces the equation to the following with the limits to cover the whole area as well, being $0 \to a$ for r' and $0 \to 2\pi$ for ϕ'

$$\mathbf{E} = \frac{\rho_s}{4\pi\epsilon_o}\iint \frac{zr'dr'd\phi'}{(r'^2+z^2)^{\frac{3}{2}}}\hat{a}_z \tag{18}$$

Now evaluating the integral and looking it up in the table will give you the following equation which is the answer.

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]\hat{a}_z \tag{19}$$

Lastly, in the limit where $z << a$ we are left with the limit for an infinite plane charge,

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o} \tag{20}$$