# Problem 1

The electric potential of a disk can be expressed in the following equation:

$$V = \frac{1}{4\pi\epsilon_o} \int \frac{\rho_s dA'}{|\mathbf{R} - \mathbf{R}'|} \tag{1}$$

For this case the vectors are the same as in HW 2, as is the $dA'$. The integrals go from $0 \to a$ for $r'$ and $0 \to 2\pi$ for $\phi'$. The integrals comes to the following:

$$V = \frac{\rho_s}{2\epsilon_o} \left( \sqrt{a^2 + z^2} - |z| \right) \tag{2}$$

# Problem 2

For this question we need to use the following relation

$$E = -\nabla V \tag{3}$$

So since the potential from problem 1 is only in the z direction this leaves just the partial with z left in the gradient. Taking the partial derivative gives the following:

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_o} \left[ 1 - \frac{z}{\sqrt{a^2 + z^2}} \right] \hat{a}_z \tag{4}$$

# Problem 3

This is in the textbook (somewhat). Note that the primary difference between your solution and that of the example is that the infinite line charge $\rho_l$ produces fields which resemble those of a trivial cylinder. That is, you now have $1/r$ dependence for the **E**-field instead of $1/r^2$, and the appearance of a natural log relationship instead of $1/r$ when comparing potentials.

# Problem 4

For this problem, we have a non-uniform permittivity and are asked to find the capacitance of a cylindrical capacitor. Fortunately, this permittivity varies only with $r$ and not with $\phi$ or $z$! Thus we can start off with the electric field from Gauss's law for a cylinder:

$$\mathbf{E} = \frac{Q}{2\pi\epsilon L r} \hat{a}_r \tag{5}$$

We assume there is a charge of $+Q$ on the inner surface and $-Q$ on the outer surface. We then integrate for the potential.

$$V = -\int_b^a \mathbf{E} \cdot d\mathbf{l} \tag{6}$$

For my notation $b$ refers to the outer radius and $a$ refers to the inner one. Plugging in our equation for $\mathbf{E}$, and noting that $d\mathbf{l} = dr\hat{a}_r$ and $\epsilon = \epsilon_o\epsilon_r$ (i.e., $\epsilon$ is now a function of $r$).

$$V = -\frac{Q}{2\pi\epsilon_o L}\int_b^a \frac{dr}{(2+\frac{4}{r})r} \tag{7}$$

Evaluating:

$$V = \frac{Q}{4\pi\epsilon_o L} ln(\frac{b+2}{a+2}) \tag{8}$$

Then using $C = Q/V$ we get the following for $C$:

$$C = \frac{4\pi\epsilon_o L}{ln(\frac{b+2}{a+2})} \tag{9}$$

Now we can plug in the values of the inner radius, outer radius, and length to obtain the capacitance: 13.26 pF (not $\mu$F, as suggested by the Cheng solutions - JBS suspects that the length was multiplied by 1000 instead of divided, thanks to ca. 1989 calculator technology).

# Problem 5

In this problem is it is important to note that the sphere is isolated, meaning we will take our potential of zero to be at infinity (JBS notes that if the sphere were "grounded" – in the electrical engineering sense and not the *Malcolm in the Middle* childhood punishment sense – then zero potential could be assigned at its surface). We can once again use Gauss's law to find the E field to set up the integral for the potential, being spherical surface and total charge Q. Our $d\mathbf{l}$ is the same as above too.

$$V = -\int_\infty^b \frac{Q}{4\pi\epsilon r^2}dr = -\int_{b+d}^b \frac{Q}{4\pi\epsilon r^2}dr - \int_\infty^{b+d} \frac{Q}{4\pi\epsilon_o r^2}dr \tag{10}$$

We can now evaluate both integrals and use the $C = Q/V$ to get the following:

$$C = 4\pi\epsilon_o\left(\frac{(b+d)^2}{d} + \chi_E\frac{b^2+bd}{d}\right) \tag{11}$$

Note that Justin Lieffers has reasonable confidence in the above simplified solution. More likely, you obtained something that is more similar in appearance to what is derived in Dr. cheng's solutions as follows:

P. 3-36 Assume charge Q on conducting sphere.

$$b < R < b+d: \quad \bar{E}_1 = \bar{a}_R \frac{Q}{4\pi\epsilon_0(1+\chi_e)R^2}.$$

$$R > b+d: \quad \bar{E}_2 = \bar{a}_R \frac{Q}{4\pi\epsilon_0 R^2}.$$

$$V = -\int_\infty^b \bar{E}\cdot d\bar{\ell} = -\int_\infty^{b+d} E_2\, dR - \int_{b+d}^{b} E_1\, dR = \frac{Q}{4\pi\epsilon_0(1+\chi_e)}\left(\frac{\chi_e}{b+d}+\frac{1}{b}\right).$$

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0(1+\chi_e)}{\frac{\chi_e}{b+d}+\frac{1}{b}}.$$

## Problem 6

For this problem it is important to note that a constant potential is being maintained across the plates so thus the charge is not the same in each region but the potential across is. We will use this as a starting point. Since the potential is equal in each region, so is the electric field, giving the following:

$$E_y = \frac{-V_o}{d} \tag{12}$$

Now, since the charge on the plates is not the same in each region, the **D**-field will not be the same in each region. We will the following relation to obtain each of them:

$$\mathbf{D} = \epsilon\mathbf{E} \tag{13}$$

Note that we aren't even mentioning **P** - Dr. Cheng did not request for us to find it! Thus,

$$\mathbf{D}_1 = \frac{-\epsilon V_o}{d}\hat{a}_y \tag{14}$$

$$\mathbf{D}_2 = \frac{-\epsilon_o V_o}{d}\hat{a}_y \tag{15}$$

Lastly, we can apply the **D**-field boundary condition at the top plate to obtain the surface charge in each region. This gives the following:

$$\rho_{s1} = \frac{\epsilon V_o}{d} \tag{16}$$

$$\rho_{s2} = \frac{\epsilon_o V_o}{d} \tag{17}$$

Note that on the bottom plate, these surface charges are equal in magnitude and opposite in sign.