Problem 1

For this problem we are given 2 hints, include integrals and that we should use, and a suggestion to take advantage of the problem's symmetry. The first step is to determine the B-field in the toroid in order to find the flux. The B-field can be obtained through Ampere's Law, which results in the following:

$$B_\phi = \frac{\mu_o NI}{2\pi(b + rcos(\theta))} \tag{1}$$

We can now use the definition of flux:

$$\Phi = \oint \mathbf{B} \cdot d\mathbf{s} \tag{2}$$

Using polar coordinates we can plug in **B** from above and use one of the integrals that was provided. We can also use symmetry to integrate over half of the circle and just double the result of the integral. The resulting Φ is the following:

$$\Phi = \mu_o NI(b - \sqrt{b^2 - a^2}) \tag{3}$$

Then, using the definition of L, we can get the following:

$$L = \mu_o N^2(b - \sqrt{b^2 - a^2}) \tag{4}$$

Problem 2

1.

a) $$H_\phi = \frac{I}{2\pi r'} = \frac{I}{2\pi[D + r\cos\phi]}$$

$$\Phi = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \mu_o H_\phi \, r dr d\phi$$

$$= \frac{\mu_o I}{2\pi} \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \frac{r dr d\phi}{[D + r\cos\phi]}$$

$$= \frac{\mu_o I}{\pi} \int_{r=0}^{a} \frac{2r}{\sqrt{D^2 - r^2}} \tan^{-1} \left\{ \frac{\sqrt{D^2 - r^2} \tan\frac{\phi}{2}}{D + r} \right\} \Bigg|_{\phi=0}^{\pi} dr$$

$$= \mu_o I \int_{r=0}^{a} \frac{r}{\sqrt{D^2 - r^2}} dr$$

$$= -\mu_o I \sqrt{D^2 - r^2} \Big|_{r=0}^{a}$$

$$= -\mu_o I \left[\sqrt{D^2 - a^2} - D\right]$$

$$M = \frac{\Phi}{I} = \mu_o \left[D - \sqrt{D^2 - a^2}\right]$$

$$R = \frac{2\pi a}{\sigma A}$$

b) $$-iR = -M\frac{dI}{dt} + L\frac{di}{dt} \rightarrow L\frac{di}{dt} + iR = M\frac{dI}{dt} \longrightarrow i = \frac{MI}{L} e^{-t/\tau} \;;\; \tau = \frac{L}{R}$$

c) $$i(t) = -\frac{MI}{L} e^{-(t - T)/\tau}$$

Problem 3

For this problem we start out with the following general formula for the energy in terms of the H-field:

$$W = \frac{1}{2}\int \mu_o H^2 dV \tag{5}$$

Then based on a uniform H-field in the toroid, as the problem states we can use the following expression for the H field inside a toroid:

$$H = \frac{N^2 I^2}{4\pi^2 b^2}, \tag{6}$$

where b is the radius to the center of toroid from the middle of the toroid. We can then simply get the volume by multiplying the area of the cross section by the circumference of the whole toroid. This results in the following, with a being the radius of the cross section:

$$W = \frac{\mu_o N^2 I^2 a^2}{4b} \tag{7}$$

We can then use the following equation for inductance from the current and the energy:

$$L = \frac{2W}{I^2} \tag{8}$$

Giving an L of:

$$L = \frac{\mu_o N^2 a^2}{2b} \tag{9}$$

Now, for the other method we first find the magnetic flux, which for a toroid of cross sectional area πa^2 and B-field of $\frac{\mu_o NI}{2\pi b}$ gives:

$$\Phi = \frac{\mu_o N I a^2}{2b} \tag{10}$$

Then using the relation between the flux and the current to get the inductance as below:

$$L = \frac{\Lambda}{I} = \frac{\Phi N}{I} \tag{11}$$

Gives the following L:

$$L = \frac{\mu_o N^2 a^2}{2b} \tag{12}$$